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The Derivation of the Quantum Kinetic Equation and the Two-Time Resolvent Method

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Van Hove's partial density matrix, $\rho_E(t)$, in his generalized master equation is interpreted as a Wigner representation of "two-time dyad" for "energy E" and "time t". This interpretation enables us to integrate the "energy" E in Van Hove's master equation. The resultant equation is of non-Markov type on two time parameters. Starting with this master equation, the derivation of quantum kinetic equations, including the second-order approximation in the density expansion, is discussed. The scaling of the quantum kinetic equation is examined in detail for a system in which particles interact through the delta shell potential. It is shown that the quantum kinetic equation, including three-particle scattering, may exist for the physical situations of low-energy scattering, high-energy scattering, and for resonance scattering for time scales of the system sufficiently separated. In deriving the quantum kinetic equation, a factorization theorem for m-particle distribution functions is proved to arbitrary order in perturbation expansion.

KEY WORDS: Van Hove's two-time method; Wigner function on energy and time; two-time dyad; Liouvillian; energy superoperator; quantum kinetic equation; factorization theorem; second-order approximation in density expansion; three-particle scattering; δ shell potential; resonance scattering.

1. INTRODUCTION

The use of generalized master equations has been central in the study of irreversible processes in nonequilibrium quantum statistical mechanics.[1-5] One formulation is the two-resolvent method of Van Hove.[1,2] In his pioneering work, he studied the time evolution of the diagonal elements of the density matrix to derive the Pauli master equation from the microscopic

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dynamical laws and extended it to a generalized master equation valid to arbitrary order in the perturbation. The generalized master equation does not, however, apply directly to the density matrix, $\rho(t)$, but to an auxiliary quantity, $\rho_E(t)$, called the partial density matrix, from which $\rho(t)$ can be obtained by energy integration.[2] This master equation has provided a viewpoint for understanding the occurrence of irreversibility in the system with infinite degrees of freedom and for justifying the phenomenological kinetic equations for macroscopic observables from the microscopic laws.

The purpose of this paper is to show first that $\rho_E(t)$ is a Wigner function[6] of "time" t and "energy" E, and to describe in full detail a derivation of the quantum kinetic equations for a moderately dense gaseous system under this interpretation.

To obtain the Wigner interpretation, in Section 2 we will show that the time t in $\rho_E(t)$ may be considered the mean time $t = (t_1 + t_2)/2$ when we express the density matrix by the dyad of the ket vector $|\psi(t_1)\rangle$ and the bra vector $\langle\psi(t_2)|$. The parameter is defined as an associated variable to the relative time $s = t_1 - t_2$, in a Fourier transform of an auxiliary two-time dyad $\rho_s(t) = \{|\psi(t + \frac{1}{2}s)\rangle\langle\psi(t - \frac{1}{2}s)|\}_{\text{av}}$, where the subscript av expresses an ensemble average of the system. We will see that the introduction of the two-time dyad enables us to establish the integration on E in Van Hove's generalized master equation. The resultant equation for $\rho_s(t)$ is a non-Markovian equation not only on the time t but also on the time s.

From this result the following question immediately arises: How may kinetic equations which include only a single parameter on time be justified through the master equation on $\rho_s(t)$ which includes two parameters on time? The second purpose of this paper is to answer this question.

As will be shown in Section 2, the answer is fairly simple when we consider the system with the limit of the lowest density or of the weak coupling interaction. Our main interest in this paper is, thus, in a more complicated case of a moderate dense system in which the second-order contribution in the density expansion cannot be neglected. In Section 3, we discuss the condition that the master equation for $\rho_s(t)$ reduces to a closed equation in the time s. By putting $s = 0$ in the closed equation, we obtain the desired master equation for the diagonal element of the density matrix with a memory effect on the time t.

In Section 4 we reduce the master equation to the quantum kinetic equations which govern a single-particle momentum distribution function, where the collision terms are represented by the T matrix for the two-body and/or the three-body scattering. In this reduction we will derive the factorization property of the distribution function [see Eq. (4.8)]; this well-known problem is encountered in the determination of the time

evolution of macroscopic observables through the formulation of the master equation. The basic idea for justifying the persistence of factorization was first given by Kac.[7] This has been rigorously discussed recently by Lanford[8] for a classical dilute system consisting of hard-sphere particles. He has discussed the derivation of the Boltzmann equation neglecting three or more particle collisions. There he proved the factorization property persists, and the iterating solution of the BBGKY hierarchy converges in the Boltzmann–Grad limit by restricting the time scale to times less than the mean free time. We must also mention Resibois' work on the classical plasma[9]; there he has sketched a proof of the persistence of factorization in perturbation theory, when he rederived the Balescu–Lenard quasilinear equation by a direct summation procedure. If one combines this result with the concept of semiconnected diagrams which has been introduced by Prigogine and Balescu,[10] the proof of the factorization property of the distribution function in kinetic equations appears to follow. This has been recently demonstrated in detail by Skarka[11] to the fourth-order approximation in perturbation theory for a classical system. We will prove this property for the quantum case less rigorously to arbitrary order in perturbation expansion in Appendix C.

In Section 5 we will discuss the physical justification (time scaling) of the quantum kinetic equation for a specific case where particles interact with the delta shell potential [see Eq. (5.1)]. We will investigate it for various cases: low energy, high energy, and resonance scattering. Then we will see that the quantum kinetic equation including the second-order approximation on the density expansion may exist for a wide class of physical situations, even for the system with resonance scattering, provided characteristic time scales are sufficiently separated.

The final section will be devoted to discussing the physical role of the two-time dyad $\rho_s(t)$, which we will introduce in Section 2, focusing our attention on the evolution of the time s. We will see that this quantity leads naturally to the criterion of dissipativity of the system and provides a plausible definition of the relaxation time of the system.

2. GENERAL THEORY

In quantum statistical mechanics, the physical state of an N-particle system is described by the density matrix $|\rho(t))$ governed by the von Neumann equation,

$$i\frac{\partial}{\partial t}|\rho(t)) = \tilde{H}|\rho(t)) \qquad (\hbar = 1) \tag{2.1}$$

with the initial condition

$$|\rho(0)) = |\{|\psi(0)\rangle\langle\psi(0)|\}_{\rm av}) \tag{2.2}$$

Here, we express the equation in a superspace, $\mathbb{S}$, with Dirac's notation, $|X)$, for the supervector (s.vector) in $\mathbb{S}$, where X is a linear operator in the ordinary Hilbert space, $\mathcal{H}$.[2] The antisymmetrized super-Hamiltonian (the Liouvillian), $\tilde{H}$, is defined through the Hamiltonian H of the system in $\mathcal{H}$ by

$$\tilde{H} = H^{>} - H^{<} \tag{2.3}$$

where the one-sided superoperators (s.operators) $A^{>}$ and $A^{<}$ are defined such that $A^{>}|B) = |AB)$ and $A^{<}|B) = |BA)$, respectively, where A is a linear operator in $\mathcal{H}$.[12]

In Eq. (2.2), $|\psi\rangle$ is a state vector of the system in $\mathcal{H}$, and the subscript av expresses an ensemble average of the system. We assume that the Hamiltonian may be decomposed into an unperturbed part and a perturbation

$$H = H_0 + V \tag{2.4}$$

We may write a formal solution of Eq. (2.1) as

$$|\rho(t)) = e^{-iH^{>}t}e^{+iH^{<}t}|\rho(0)) = |\{|\psi(t)\rangle\langle\psi(t)|\}_{\rm av}) \tag{2.5}$$

In order to investigate the long-time behavior of this formal solution based on Van Hove's two-resolvent method,[2] we will see that it is convenient to introduce an auxiliary dyad which depends on two time parameters, t and s:

$$\begin{aligned} |\rho_s(t)) &= e^{-iH^{>}[t+(1/2)s]}e^{+iH^{<}[t-(1/2)s]}|\rho(0)) \\ &= |\{|\psi(t+\tfrac{1}{2}s)\rangle\langle\psi(t-\tfrac{1}{2}s)|\}_{\rm av}) \end{aligned} \tag{2.6}$$

By putting $s = 0$, the two-time dyad reduces to the ordinary density matrix. The trace of $\rho_s(t)$ in $\mathcal{H}$ expresses the ensemble average of the surviving amplitude of the system between two different times,

$$\mathrm{tr}\,\rho_s(t) = \{\langle\psi(t)\,|\,\psi(t+s)\rangle\}_{\rm av} \tag{2.7}$$

and, thus, it is essentially the same for the two-time Green function.[13] We will further examine some asymptotic properties of $\rho_s(t)$ in Section 6.

[2] The inner product in $\mathbb{S}$ is defined by $(X|Y) = \mathrm{tr}[X^{+}Y]$, where tr means the trace and X^{+} is the adjoint operator of X in $\mathcal{H}$ (see Ref. 12 for more details).

Let us express the time evolution in Eq. (2.6) as

$$\mathfrak{U}_s(t) = e^{-iH^{>}[t+(1/2)s]}e^{+iH^{<}[t-(1/2)s]} \tag{2.8}$$

This evolution s.operator satisfies the following equations:

$$i\frac{\partial}{\partial t}\mathfrak{U}_s(t) = \tilde{H}\mathfrak{U}_s(t) = \mathfrak{U}_s(t)\tilde{H} \tag{2.9a}$$

$$i\frac{\partial}{\partial s}\mathfrak{U}_s(t) = \hat{H}\mathfrak{U}_s(t) = \mathfrak{U}_s(t)\hat{H} \tag{2.9b}$$

with the initial condition

$$\mathfrak{U}_0(0) = 1 \tag{2.10}$$

where the symmetrized super-Hamiltonian (the energy s.operator), $\hat{H}$, is defined by[12,15]

$$\hat{H} = \tfrac{1}{2}(H^{>} + H^{<}) \tag{2.11}$$

Both super-Hamiltonians $\tilde{H}$ and $\hat{H}$ are Hermitian in $\mathbb{S}$ and commute with each other:

$$\tilde{H}\hat{H} = \hat{H}\tilde{H} \tag{2.12}$$

The evolution s.operator $\mathfrak{U}_s(t)$ is expressed by the resolvent operator of the Hamiltonian $R(z) = (H - z)^{-1}$ as

$$\mathfrak{U}_s(t) = \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma dz \int_{\Gamma'} dz'\, e^{-iz[t+(1/2)s]}e^{+iz'[t-(1/2)s]}R^{>}(z)R^{<}(z') \tag{2.13}$$

where the paths of integration Γ and Γ' are any positive contours enclosing sufficiently large portions of the real axes in the complex z and z' planes, respectively. For the case of $t > \frac{1}{2}|s|$, only the path $\Gamma(\Gamma')$ in the upper (lower) half-plane contributes. By changing the variables z and z' to $E = (z + z')/2$ and $l = z - z'$, Eq. (2.13) is rewritten

$$\mathfrak{U}_s(t) = \int_{-\infty}^{+\infty} dE\, e^{-iEs}\mathfrak{U}_E(t), \qquad \text{for} \quad t > \tfrac{1}{2}|s| \tag{2.14}$$

where

$$\mathfrak{U}_E(t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma''} dl\, e^{-ilt}R^{>}(E + \tfrac{1}{2}l)R^{<}(E - \tfrac{1}{2}l) \tag{2.15}$$

and the path Γ'' is parallel to the real axis on the upper half-plane in the complex l plane and goes to $-\infty$ from $+\infty$.

We now introduce a quantity

$$|\rho_E(t)) = \mathfrak{U}_E(t)|\rho(0)) \tag{2.16}$$

It is related to the two-time dyad through the Fourier transform of the relative time s in Eq. (2.6):

$$|\rho_s(t)) = \int_{-\infty}^{+\infty} dE\, e^{-iEs} |\rho_E(t)) \tag{2.17}$$

Therefore, $|\rho_E(t))$ is a Wigner distribution function on "time t" and "energy E." If we put $s = 0$ in the above definitions, they reduce to the quantities which were introduced by Van Hove, and $|\rho_E(t))$ is then just the partial density matrix.[2]

In order to derive a quantum kinetic equation, we start with the asymptotic master equation which governs an asymptotic time evolution of $|\rho_E(t))$. The derivation of the master equation has been discussed in Refs. (12) and (14). There we have shown that an analogous discussion to the well-known one-time formalism developed by Prigogine and coworkers[3–5] can be applied to Van Hove's two-time formalism, if we decompose the perturbation series of the product of two resolvents, $R^>(E + \frac{1}{2}l)$ $R^<(E - \frac{1}{2}l)$, into the following four fundamental components: a creation part, a diagonal part, a destruction part, and a propagation-of-correlation part, which are represented by fundamental kinetic s.operators, respectively. To avoid lengthy redefinitions of these s.operators and to go directly to the main theme of this article, we review the definitions and important results of this decomposition in Appendix A, and we write only the resultant asymptotic master equations [see Eq. (A.21)]:

$$i\frac{\partial}{\partial t} P\Sigma_E(t)P = 2\pi \int_0^\infty dt'\, \chi'_E(t') P\Sigma_E(t - t')P \tag{2.18a}$$

$$i\frac{\partial}{\partial t} P\Sigma_E(t)P = 2\pi \int_0^\infty dt'\, P\Sigma_E(t - t')P\tilde{\chi}'_E(t')P \tag{2.18b}$$

Here, $\Sigma_E(t)$ is the asymptotic evolution part of $\mathfrak{A}_E(t)$, which is obtained by evaluating the contribution of the poles at $l = +i0$ in the perturbation series of $R^>(E + \frac{1}{2}l)R^<(E - \frac{1}{2}l)$ in Eq. (2.15). The s.operator P is the projection s.operator [see Eq. (A.3)] which projects out diagonal components of the s.vector on the eigenstate of H_0. The kinetic operators, χ'_E and $\tilde{\chi}'_E$, are called the "collision s.operators." They describe the irreducible diagonal transition between two diagonal s.states in the perturbation expansion. This transition occurs in the off-diagonal subspace. The collision s.operators are defined in the Laplace transform by $\chi_E(l) = \Delta_E(l)\mathfrak{W}_E(l) - \mathcal{G}_E(l)$ and $\tilde{\chi}_E(l) = \mathfrak{W}_E(l)\Delta_E(l) - \mathcal{G}_E(l)$ [see also Eqs. (A.16) and (A.22)], where $\mathfrak{W}_E$ and $\mathcal{G}_E$ are irreducible diagonal s.operators and Δ_E is the difference of the renormalized propagator [see Eqs. (A.9)–(A.17)]. The collision s.operators are generalizations of the collision kernel of the Pauli

master equation to arbitrary order in the perturbation expansion. Similar to the Pauli master equation, the collision s.operators consist of two parts; the gain part, $\Delta_E \mathfrak{W}_E$ or $\mathfrak{W}_E \Delta_E$, and the loss part, $\mathcal{G}_E$ [see Eq. (A.16)]. The basic assumption which we have imposed in deriving Eq. (2.18) is that the kinetic s.operators are regular functions on l in the neighborhood of $l = +i0$. The validity of this fundamental assumption will be discussed in Section 5 by using specific examples.

We introduce an asymptotic two-time evolution s.operator, $\Sigma_s(t)$, as a Fourier transform of $\Sigma_E(t)$,

$$\Sigma_s(t) = \int_{-\infty}^{+\infty} dE\, e^{-iEs} \Sigma_E(t) \tag{2.19}$$

Then, the Fourier transform of the master equations (2.18) is expressed in the form of convolution integrals on t and s as

$$i \frac{\partial}{\partial t} P\Sigma_s(t)P = \int_0^{\infty} dt' \int_{-\infty}^{\infty} ds'\, \chi_{s'}''(t') P\Sigma_{s-s'}(t-t')P \tag{2.20a}$$

$$i \frac{\partial}{\partial t} P\Sigma_s(t)P = \int_0^{\infty} dt' \int_{-\infty}^{+\infty} ds'\, P\Sigma_{s-s'}(t-t')P\tilde{\chi}_{s'}''(t') \tag{2.20b}$$

where

$$\chi_s''(t) = \int_{-\infty}^{+\infty} dE\, e^{-iEs} \chi_E'(t) \tag{2.21a}$$

$$\tilde{\chi}_s''(t) = \int_{-\infty}^{+\infty} dE\, e^{-iEs} \tilde{\chi}_E'(t) \tag{2.21b}$$

The master equations (2.20) have memory effects not only on t but also on s. By putting $s = 0$, the evolution s.operator $\Sigma_s(t)$ describes the asymptotic time evolution of the ordinary density matrix $|\rho(t))$. Equations (2.20) show clearly that there is no closed master equation for $P|\rho(t))$ itself as asserted by Van Hove.[2]

In the derivation of the quantum kinetic equation, it is important to consider what condition allows the closure of the master equation on s. To find the condition, let us write the right-hand side of Eq. (2.20a) in terms of $\chi_E'(t)$ and expand in a Taylor series around the regular point $E = \omega$:

$$\begin{aligned} &\int_0^{\infty} dt' \int_{-\infty}^{+\infty} dE'\, e^{-iEs} \chi_E'(t') P\Sigma_E(t-t')P \\ &\quad = \int_0^{\infty} dt' \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial^n}{\partial E^n} \chi_E'(t') \right]_{E=\omega} \\ &\qquad \times \int_{-\infty}^{+\infty} dE\, e^{-iEs} (E-\omega)^n P\Sigma_E(t-t')P \end{aligned} \tag{2.22}$$

This shows that if $P\Sigma_E(t)P$ contains a factor which has a sharp peak around $E = \omega$ [$\delta(E - \omega)$-like], then Eqs. (2.20) may close at the time s.

For the limits of weak coupling or low density, it is easy to see that the δ singularity comes from the factor $(\alpha; \alpha|\Delta_E(+i0)|\alpha; \alpha) = 2\pi i\delta(E - \epsilon_\alpha)$ in the solution of $\Sigma_E(t)$ [see (A.20)], where ϵ_α is an unperturbed energy, i.e., $H_0|\alpha\rangle = \epsilon_\alpha|\alpha\rangle$, and we can obtain a closed master equation (for example, the Pauli master equation[1]). Our main interest here is that of more complicated systems in which higher-order terms of the density expansion or the coupling constant cannot be neglected. In the following section, we will discuss a moderate dense system where the second-order term of the density expansion cannot be neglected. Thus, we will be concerned with both binary and three-body scattering in the quantum kinetic equation.[16,17][3] The system with moderately strong coupling can be treated in a similar way. The situation of really strong coupling (and high density) still remains beyond our understanding.

3. THE DENSITY EXPANSION AND THE δ SINGULARITY

We will discuss the origin of the δ singular factor in the second-order approximation of the density expansion. Because the kinetic s.operators, χ_E, $\tilde{\chi}_E$, $Q\mathcal{C}_E P$, and $P\mathcal{D}_E Q$ in Eq. (A.20) start with the first-order term on the density c, and Δ_E starts with the zeroth-order term, the solution of $\Sigma_E(t)$ is expressed from (A.20)–(A.26) in this approximation by

$$\Sigma_E(t) \approx [P + Q\mathcal{C}_E P]\Sigma_E(t)[P + P\mathcal{D}_E Q] \tag{3.1}$$

and

$$P\Sigma_E(t)P = e^{-iP\Gamma_E Pt}P\Sigma_E(0)P = P\Sigma_E(0)Pe^{-iP\tilde{\Gamma}_E Pt} \tag{3.2}$$

where

$$P\Gamma_E P \approx (1 + \partial_I \chi_E)\chi_E \tag{3.3a}$$

$$P\tilde{\Gamma}_E P \approx \tilde{\chi}_E(1 + \partial_I \tilde{\chi}_E) \tag{3.3b}$$

and

$$P\Sigma_E(0)P \approx \frac{1}{2\pi i}\left[\Delta_E + \partial_I(\chi_E \Delta_E)\right] \tag{3.4a}$$

$$= \frac{1}{2\pi i}\left[\Delta_E + \partial_I(\Delta_E \tilde{\chi}_E)\right] \tag{3.4b}$$

[3] The product of the one-particle functions, $\phi_{1,E}$, in Eqs. (3.1), (3.9), and (3.11) in Ref. 15 should be read as the convolution integral of E.

where $\partial_l \equiv \partial/\partial l$, and hereafter all functions of l are evaluated at $l = +i0$. In Eqs. (3.3), the factors $\partial_l \chi_E$ and $\partial_l \tilde{\chi}_E$ appear as a result of the memory effect of t, i.e., the non-Markovian structure of the generalized master equation on t [see Eqs. (2.20) and also (A.21)–(A.24)]. From Eq. (3.4b) we have

$$P\Sigma_E(0)P = \frac{1}{2\pi i}\Delta_E(1 + \partial_l \tilde{\chi}_E) + \frac{1}{2\pi i}(\partial_l \Delta_E)\tilde{\chi}_E \tag{3.5}$$

We now show that the δ-singular factor comes from Δ_E in the first term in Eq. (3.5), while the second term can be neglected. Let us consider the tetradic element of Δ_E [see (A.9) and (A.14)]:

$$\begin{aligned}\frac{1}{2\pi i}(\alpha;\alpha|\Delta_E|\alpha;\alpha) &= \frac{1}{2\pi i}\left[D_\alpha(E + i0) - D_\alpha(E - i0)\right] \\ &= \frac{1}{\pi}\frac{\nu_2 J_\alpha(E)}{\left[\epsilon_\alpha - E - \nu_1 K_\alpha(E)\right]^2 + \left[\nu_2 J_\alpha(E)\right]^2}\end{aligned} \tag{3.6}$$

where $\nu_1 K_\alpha$ and $\nu_2 J_\alpha$ are the real and imaginary parts of the self-energy part $G_\alpha(z)$ at the real axis of z,[2]

$$G_\alpha(E \pm i0) = \nu_1 K_\alpha(E) \pm i\nu_2 J_\alpha(E) \tag{3.7}$$

and ν_1 and ν_2 are dimensionless parameters which characterize the magnitude of the real part and imaginary part, respectively.

Now we make the basic assumption that the imaginary part, $\nu_2 J_\alpha$, does not vanish, but is so small that the function (3.6) has a sufficiently sharp peak at the point of $E = E_\alpha$, where E_α is a solution of the equation

$$E_\alpha - \epsilon_\alpha + \nu_1 K_\alpha(E_\alpha) = 0 \tag{3.8}$$

As will be discussed in Section 6, the first part of this assumption, i.e., $\nu_2 J_\alpha \neq 0$, is essential to obtain the dissipative behavior of the system.

Under this assumption we can approximate the functions $\epsilon_\alpha - E - \nu_1 K_\alpha(E)$ and $\nu_2 J_\alpha(E)$ in Eq. (3.6) by a Taylor series around at $E = E_\alpha$ to first order. Neglecting higher-order terms on ν_2, we have

$$\frac{1}{2\pi i}(\alpha;\alpha|\Delta_E|\alpha;\alpha) \approx \frac{n_\alpha}{\pi}\frac{\gamma_\alpha}{(E - E_\alpha)^2 + \gamma_\alpha^2} + \frac{n_\alpha}{\pi}\frac{J'_\alpha(E_\alpha)}{J_\alpha(E\alpha)}\frac{(E - E_\alpha)\gamma_\alpha}{(E - E_\alpha)^2 + \gamma_\alpha^2} \tag{3.9}$$

with

$$n_\alpha \equiv \frac{1}{1 + \nu_1 K'_\alpha(E\alpha)}, \qquad \gamma_\alpha \equiv n_\alpha \nu_2 J_\alpha(E_\alpha) > 0 \tag{3.10}$$

where n_α is a renormalization constant and prime on K_α and J_α expresses a derivative on E. The quantity γ_α gives a width of the function (3.9) around

$E = E_\alpha$. If the condition, $|\gamma_\alpha / E_\alpha| \ll 1$, i.e.,

$$\left| \frac{n_\alpha \nu_2 J_\alpha(E_\alpha)}{\epsilon_\alpha - \nu_1 K_\alpha(E_\alpha)} \right| \ll 1 \tag{3.11}$$

is fulfilled, and if the remaining functions in Eq. (3.2) have no singularity near $E = E_\alpha$, then we can approximate the first term in Eq. (3.9) by the δ function, and the second term vanishes because of $(E - E_\alpha)\delta(E - E_\alpha) = 0$. Thus, we have

$$\frac{1}{2\pi i}(\alpha;\alpha|\Delta_E|\alpha;\alpha) \approx n_\alpha \delta(E - E_\alpha) \tag{3.12}$$

In a similar approximation to Eq. (3.9), we have for the second term in Eq. (3.5)

$$\begin{aligned} &\frac{1}{2\pi i}(\alpha;\alpha|\partial_I \Delta_E|\alpha;\alpha)(\alpha;\alpha|\tilde{\chi}_E|\beta;\beta) \\ &\approx \left\{ \frac{1}{\pi} \left[\frac{(E - E_\alpha)^2 \gamma_\alpha}{\left[(E - E_\alpha)^2 + \gamma_\alpha^2\right]^2} - \frac{\gamma_\alpha^3}{\left[(E - E_\alpha)^2 + \gamma_\alpha^2\right]^2} \right] \right. \\ &\quad \left. + \frac{n_\alpha}{\pi} \frac{J'_\alpha(E_\alpha)}{J_\alpha(E_\alpha)} \frac{2(E - E_\alpha)\gamma_\alpha^3}{\left[(E - E_\alpha)^2 + \gamma_\alpha^2\right]^2} \right\} \\ &\quad \times \left[\frac{1}{2i} \frac{(\alpha;\alpha|\mathfrak{W}_{E_\alpha} \Delta_{E_\alpha}|\beta;\beta)}{\nu_2 J_\alpha(E_\alpha)} - \delta_{\alpha,\beta} \right] \end{aligned} \tag{3.13}$$

Condition (3.11) shows that the first term in the curly bracket can be approximated by $\delta(E - E_\alpha)/2 - \delta(E - E_\alpha)/2$, while the second term by $n_\alpha(J'_\alpha/J_\alpha)(E - E_\alpha)\delta(E - E_\alpha)$.[18] Therefore, if we assume that the gain part of the collision operator, $\mathfrak{W}_{E_\alpha} \Delta_{E_\alpha}$, has the same order of magnitude as the loss part, $\nu_2 J_\alpha(E_\alpha)$ [see Eqs. (A.16b) and (A.15)], then we can neglect Eq. (3.13) as compared to the first term in Eq. (3.5). Thus, we obtain

$$P\Sigma_E(0)P \approx \frac{1}{2\pi i} \Delta_E (1 + \partial_I \tilde{\chi}_E) \tag{3.14a}$$

In a similar way in Eq. (3.4a) we obtain

$$P\Sigma_E(0)P \approx \frac{1}{2\pi i}(1 + \partial_I \chi_E)\Delta_E \tag{3.14b}$$

with Eq. (3.12). Combining these results with Eq. (3.2), we conclude that $P\Sigma_E(t)P$ contains a δ-singular factor, if the condition (3.11) and the analytic properties of the kinetic s.operators which we imposed above are satisfied. We can thus obtain a closed master equation for the diagonal element of the ordinary density matrix.

Substituting Eqs. (3.1)–(3.3) and Eq. (3.14) with (3.12) into Eq. (2.19), and taking a time derivative on t and operating it to $|\rho(0))$, we have

$$i\frac{\partial}{\partial t}(\alpha;\alpha|\rho_s^{\mathrm{as}}(t)) = \sum_{\beta}(\alpha;\alpha|(1+\partial_l\chi_{E_\beta})\chi_{E_\beta}|\beta;\beta)(\beta;\beta|\rho_s^{\mathrm{as}}(t)) \quad (3.15)$$

where the asymptotic two-time dyad is defined by

$$|\rho_s^{\mathrm{as}}(t)) = \Sigma_s(t)|\rho(0)) \quad (3.16)$$

Finally let us comment: this master equation [Eq. (3.15)] is valid for the asymptotic time limit $t\to\infty$ in a moderate dense system in which we keep the terms in the perturbation expansion which are proportional to $(ct)^m$ and $(c^2t)^n$, where m and n are positive integers. The basic assumptions in obtaining the closed master equation (3.15) in the relative time s are the sharpness condition for the function $(\alpha;\alpha|\Delta_E|\alpha;\alpha)$, i.e., Eq. (3.11), and the analyticity of the remaining functions in Eq. (3.2) at $E = E_\alpha$. The physical justification of these assumptions will be discussed for a specific example in Section 5.

4. THE QUANTUM KINETIC EQUATIONS

We now reduce our master equation (3.15) to the quantum kinetic equation for a single-particle momentum distribution function, which includes an effect of three-body scattering expressed by a T matrix.[19]

Using the relations (A.16a) and (A.18) in Eq. (3.15) and neglecting terms proportional to $(\partial_l\Delta_E)\mathfrak{W}_E$ and $\mathfrak{W}_E(\partial_l\Delta_E)$ by a similar argument as in Eq. (3.13), we have, for $s = 0$,

$$\begin{aligned}
&i\frac{\partial}{\partial t}(\alpha;\alpha|\rho_0^{\mathrm{as}}(t)) \\
&\quad = \sum_{\beta}\big[(\alpha;\alpha|\Delta_{E_\beta}\mathfrak{W}_{E_\beta}|\beta;\beta)(\beta;\beta|\rho_0^{\mathrm{as}}(t)) \\
&\qquad - (\alpha;\alpha|\mathfrak{W}_{E_\alpha}\Delta_{E_\alpha}|\beta;\beta)(\alpha;\alpha|\rho_0^{\mathrm{as}}(t))\big] \\
&\quad + \sum_{\beta}\sum_{\gamma}\big[(\alpha;\alpha|\Delta_{E_\beta}(\partial_l\mathfrak{W}_{E_\beta})|\gamma;\gamma)(\gamma;\gamma|\Delta_{E_\beta}\mathfrak{W}_{E_\beta}|\beta;\beta)(\beta;\beta|\rho_0^{\mathrm{as}}(t)) \\
&\qquad - (\alpha;\alpha|\Delta_{E_\gamma}(\partial_l\mathfrak{W}_{E_\gamma})|\gamma;\gamma)(\gamma;\gamma|\mathfrak{W}_{E_\gamma}\Delta_{E_\gamma}|\beta;\beta)(\gamma;\gamma|\rho_0^{\mathrm{as}}(t)) \\
&\qquad - (\alpha;\alpha|(\partial_l\mathfrak{W}_{E_\beta})\Delta_{E_\beta}|\gamma;\gamma)(\alpha;\alpha|\Delta_{E_\beta}\mathfrak{W}_{E_\beta}|\beta;\beta)(\beta;\beta|\rho_0^{\mathrm{as}}(t)) \\
&\qquad + (\alpha;\alpha|(\partial_l\mathfrak{W}_{E_\alpha})\Delta_{E_\alpha}|\gamma;\gamma)(\alpha;\alpha|\mathfrak{W}_{E_\alpha}\Delta_{E_\alpha}|\beta;\beta)(\alpha;\alpha|\rho_0^{\mathrm{as}}(t))\big]
\end{aligned} \quad (4.1)$$

This equation still contains arbitrary order of the density c, because the dressed energy, E_α, Eq. (3.8), is a function of $\nu_1 \propto c$. In order to obtain an equation through the second order of c, we expand all functions of the dressed energy around the unperturbed energy and drop higher-order terms of the density than c^2. Then, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}(\alpha;\alpha|\rho_0^{\mathrm{as}}(t)) &= 2\pi\sum_\beta \delta(\epsilon_\alpha-\epsilon_\beta)(\alpha;\alpha|\mathcal{W}^{(2)}_{\epsilon_\alpha}|\beta;\beta) \\
&\quad\times[(\beta;\beta|\rho_0^{\mathrm{as}}(t))-(\alpha;\alpha|\rho_0^{\mathrm{as}}(t))] \\
&\quad+2\pi\sum_\beta \delta(\epsilon_\alpha-\epsilon_\beta)(\alpha;\alpha|\mathcal{W}^{(3)}_{\epsilon_\alpha}|\beta;\beta) \\
&\quad\times[(\beta;\beta|\rho_0^{\mathrm{as}}(t))-(\alpha;\alpha|\rho_0^{\mathrm{as}}(t))] \\
&\quad-2\pi\sum_\beta\{\delta'(\epsilon_\alpha-\epsilon_\beta)(\alpha;\alpha|\mathcal{W}^{(2)}_{\epsilon_\alpha}|\beta;\beta) \\
&\qquad\times[K_\alpha^{(2)}(\epsilon_\alpha)-K_\beta^{(2)}(\epsilon_\beta)] \\
&\qquad+\delta(\epsilon_\alpha-\epsilon_\beta)(\alpha;\alpha|\mathcal{W}^{(2)\prime}_{\epsilon_\alpha}|\beta;\beta)K_\alpha^{(2)}(\epsilon_\alpha) \\
&\qquad+\delta(\epsilon_\alpha-\epsilon_\beta)(\alpha;\alpha|\mathcal{W}^{(2)}_{\epsilon_\alpha}|\beta;\beta)K_\alpha^{(2)\prime}(\epsilon_\alpha)\} \\
&\quad\times[(\beta;\beta|\rho_0^{\mathrm{as}}(t))-(\alpha;\alpha|\rho_0^{\mathrm{as}}(t))] \\
&\quad+(2\pi)^2\sum_\beta\sum_\gamma \delta(\epsilon_\alpha-\epsilon_\gamma)\delta(\epsilon_\gamma-\epsilon_\beta)(\alpha;\alpha|\partial_t\mathcal{W}^{(2)}_{\epsilon_\alpha}|\gamma;\gamma) \\
&\quad\times\{(\gamma;\gamma|\mathcal{W}^{(2)}_{\epsilon_\gamma}|\beta;\beta)[(\beta;\beta|\rho_0^{\mathrm{as}}(t))-(\gamma;\gamma|\rho_0^{\mathrm{as}}(t))] \\
&\qquad-(\alpha;\alpha|\mathcal{W}^{(2)}_{\epsilon_\alpha}|\beta;\beta)[(\beta;\beta|\rho_0^{\mathrm{as}}(t))-(\alpha;\alpha|\rho_0^{\mathrm{as}}(t))]\}
\end{aligned} \tag{4.2}$$

where the superscript (n) indicates that the quantity is a function of n particles.

The tetradic element of $\mathcal{W}^{(2)}_{\epsilon_\alpha}$ and $\mathcal{W}^{(2)}_{\epsilon_\alpha}$ can be rewritten by using the

two-body and three-body T matrices defined by[16]

$$t_{ij}(z) = V_{ij} + V_{ij}\frac{1}{z - H_{ij}^0}t_{ij}(z), \qquad i, j = 1,2,3 \tag{4.3}$$

and

$$T(z) = \sum_{i<j}^{3}\left[V_{ij} + V_{ij}\frac{1}{z - H_{123}^0}T(z)\right] \tag{4.4}$$

where H_{ij}^0 and H_{123}^0 are the two-body and three-body unperturbed Hamiltonians, respectively, and

$$\langle \mathbf{p}_i, \mathbf{p}_j | V_{ij} | \mathbf{p}_i', \mathbf{p}_j' \rangle = \frac{1}{(2\pi)^3\Omega}\int d\mathbf{r}\, V(r) e^{i(\mathbf{p}_i - \mathbf{p}_i')\mathbf{r}}\delta_{\mathbf{p}_i + \mathbf{p}_j, \mathbf{p}_i' + \mathbf{p}_j'}$$

$$= \frac{1}{\Omega}v(|\mathbf{p}_i - \mathbf{p}_i'|)\delta_{\mathbf{p}_i + \mathbf{p}_j, \mathbf{p}_i' + \mathbf{p}_j'} \tag{4.5}$$

and $(2\pi)^3\Omega$ is the volume of the system.

The N-particle momentum distribution function $\phi^{(N)}$ and its reduced function $\phi^{(m)}$ are related to the density matrix by

$$\phi^{(N)}(\mathbf{p}_1, \ldots, \mathbf{p}_N; t) = (\alpha; \alpha | \rho_0(t)) \tag{4.6}$$

and

$$\phi^{(m)}(\mathbf{p}_1, \ldots, \mathbf{p}_m; t) = \Omega^{-N+m}\sum_{\mathbf{p}_{m+1}} \cdots \sum_{\mathbf{p}_N}\phi^{(N)}(\mathbf{p}_1, \ldots, \mathbf{p}_N; t) \tag{4.7}$$

where $|\alpha\rangle = |\mathbf{p}_1, \ldots, \mathbf{p}_N\rangle$.

To obtain the kinetic equation for a single-particle distribution function from the master equation, we need to prove the factorization property of the distribution function of the finite m particles,

$$\phi^{(m)}(\mathbf{p}_1, \ldots, \mathbf{p}_m; t) = \prod_{r=1}^{m}\phi^{(1)}(\mathbf{p}_r; t) \tag{4.8}$$

The essential assumptions in the derivation are (a) the thermodynamic limit,

$$N \to \infty, \qquad \Omega \to \infty, \qquad N/\Omega = (2\pi)^3 c = \text{const} \tag{4.9}$$

and (b) the initial condition that the spatial correlation among particles vanishes when a distance between particles increases. The proof of the factorization property (4.8) is given in Appendix C.

In Eq. (4.2), summing over all momenta except $\mathbf{p}_1$ of the fixed particle, and expressing $\mathfrak{V}_{\epsilon_\alpha}^{(n)}$ by the T matrices, and using the factorization property

(4.8), we obtain the quantum kinetic equation,

$$\begin{aligned}
\frac{\partial}{\partial t}\phi(1) = {} & (2\pi)^3 c \sum_{\mathbf{p}_2} \sum_{\mathbf{p}_1'\mathbf{p}_2'} 2\pi\delta(\epsilon_1 + \epsilon_2 - \epsilon_{1'} - \epsilon_{2'}) |t^+_{12,1'2'}(\epsilon_1 + \epsilon_2)|^2 \\
& \times \left[\phi(1')\phi(2') - \phi(1)\phi(2)\right] \\
& + \left[(2\pi)^3 c\right]^2 \sum_{\mathbf{p}_2\mathbf{p}_3} \sum_{\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'} 2\pi\delta(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_{1'} - \epsilon_{2'} - \epsilon_{3'}) \\
& \times \left\{ T^+_{123,1'2'3'}(\epsilon_1 + \epsilon_2 + \epsilon_3) T^-_{1'2'3',123}(\epsilon_1 + \epsilon_2 + \epsilon_3) \right\}_{\text{t.ti}} \\
& \times \left[\phi(1')\phi(2')\phi(3') - \phi(1)\phi(2)\phi(3)\right] \\
& - \frac{1}{2}\left[(2\pi)^3 c\right]^2 \sum_{\mathbf{p}_2\mathbf{p}_3} \sum_{P(123)} \sum_{\mathbf{p}_1'\mathbf{p}_2'} \\
& \times \left\{ 2\pi \frac{\partial\delta(E - \epsilon_1' - \epsilon_2')}{\partial E}\bigg|_{E=\epsilon_1+\epsilon_2} |t^+_{12,1'2'}(\epsilon_1 + \epsilon_2)|^2 \right. \\
& \quad \times \left[K_{13}(\epsilon_1 + \epsilon_3) - K_{1'3}(\epsilon_{1'} + \epsilon_3)\right] \\
& \quad + 2\pi\delta(\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_2') \frac{\partial |t^+_{12,1'2'}(E)|^2}{\partial E}\bigg|_{E=\epsilon_1+\epsilon_2} K_{13}(\epsilon_1 + \epsilon_3) \\
& \quad \left. + 2\pi\delta(\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_2') |t^+_{12,1'2'}(\epsilon_1 + \epsilon_2)|^2 \frac{\partial K_{13}(E)}{\partial E}\bigg|_{E=\epsilon_1+\epsilon_3} \right\} \\
& \times \left[\phi(1')\phi(2')\phi(3) - \phi(1)\phi(2)\phi(3)\right] \\
& + \frac{1}{2}\left[(2\pi)^3 c\right]^2 \sum_{\mathbf{p}_2\mathbf{p}_3} \sum_{P(123)} \sum_{\mathbf{p}_1'\mathbf{p}_2'} \sum_{\mathbf{p}_2''\mathbf{p}_3''} 2\pi^2\delta(\epsilon_1 + \epsilon_3 - \epsilon_1' - \epsilon_2') \\
& \times \delta(\epsilon_1' + \epsilon_2 - \epsilon_1'' - \epsilon_3'') \\
& \times \left[\frac{\partial t^+_{13,1'3'}(E)}{\partial E} t^-_{13,1'3'}(E) - t^+_{13,1'3'}(E) \frac{\partial t^-_{13,1'3'}(E)}{\partial E} \right]_{E=\epsilon_1+\epsilon_3} \\
& \times \left\{ |t^+_{1'2,1''2''}(\epsilon_1' + \epsilon_2)|^2 \left[\phi(1'')\phi(2'')\phi(3') - \phi(1')\phi(2)\phi(3')\right] \right. \\
& \quad \left. - |t^+_{12,1''2''}(\epsilon_1 + \epsilon_2)|^2 \left[\phi(1'')\phi(2'')\phi(3') - \phi(1)\phi(2)\phi(3)\right] \right\}
\end{aligned} \tag{4.10}$$

where the subscript t.ti stands for "two-side topologically irreducible" [see

Appendix A] and we use abbreviated notation $\epsilon_i \equiv \epsilon_{\mathbf{p}_i}$, $\phi(j') \equiv \phi^{(1)}(\mathbf{p}'_j; t)$, and

$$\begin{aligned} t^{\pm}_{ij,i'j'}(E) &\equiv \langle \mathbf{p}_i, \mathbf{p}_j | t_{ij}(E \pm i0) | \mathbf{p}'_i, \mathbf{p}'_j \rangle \\ T^{\pm}_{123,1'2'3'}(E) &\equiv \langle \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 | T(E \pm i0) | \mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3 \rangle \end{aligned} \tag{4.11}$$

and $K_{ij}(E)$ is a real part of the forward scattering amplitude which arises from the self-energy part, $G_\alpha(E \pm i0)$,

$$K_{ij}(E) \equiv -\mathrm{Re}\left[t^{+}_{ij,ij}(E) \right] \tag{4.12}$$

The summation, $\sum_{P(123)}$, in Eq. (4.10) is taken over all permutation of particles in its argument. The factor $1/2$ in the second and third terms arises from the fact that dummy particles 2 and 3 are counted twice. This result agrees with the equation derived by P. Resibois[17] and later by F. Mayne[20] by using the one-resolvent method.

The first term (the Boltzmann term) in the right-hand side in Eq. (4.10) comes from the two-body scattering. The second term comes from the purely three-body scattering. The third term arises from the effects of background particles surrounding the scattering, and it is a typical effect of many-particle systems. It consists of two contributions: the first two terms in the curly bracket come from a dressing effect of the energy, while the last term comes from the effect of the state renormalization. The last term in Eq. (4.10) is due to the memory effect on the time t.

In the following section we turn to a specific example to illustrate the physical time scaling for justifying the derivation of Eq. (4.10).

5. VALIDITY OF THE QUANTUM KINETIC EQUATION—TIME SCALING

We will discuss the validity of Eq. (4.10) for a system of particles which interact with each other through the δ shell potential of radius and strength λ,

$$V(r) = -\frac{\lambda}{m} \delta(r - a) \tag{5.1}$$

where m is the mass of a particle and λa is a dimensionless parameter.[21] This potential has the following simple properties: (i) we can obtain the off-shell two-body T matrix exactly, (ii) it exhibits resonance scattering, and (iii) in the limit of $\lambda a \to \infty$, the T matrix reduces to the T matrix for the hard-sphere potential of radius a.

Let us write down the T matrix explicitly. In order to take into account the conservation of momenta in a scattering, we introduce a relative

two-body coordinate:

$$\mathbf{p} = (\mathbf{p}_1 - \mathbf{p}_2)/2, \qquad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$
$$\mathbf{p}' = (\mathbf{p}'_1 - \mathbf{p}'_2)/2, \qquad \mathbf{P}' = \mathbf{p}'_1 + \mathbf{p}'_2 \tag{5.2}$$

Putting

$$t_{12,1'2'}(z) = t_{\mathbf{pp}'}(\omega)\delta_{\mathbf{P},\mathbf{P}'} \tag{5.3}$$

we get from Eq. (4.3)

$$t_{|\mathbf{pp}'|}(\omega) = \Omega^{-1} v(|\mathbf{p} - \mathbf{p}'|) + \Omega^{-1} \sum_{\mathbf{p}''} v(|\mathbf{p} - \mathbf{p}''|) \frac{1}{\omega - (\mathbf{p}''^2/2\mu)} t_{\mathbf{p}''\mathbf{p}}(\omega) \tag{5.4}$$

where a new parameter, $\omega \equiv z - (P^2/2M)$, is introduced, and $\mu \equiv m/2$ and $M \equiv 2m$ are reduced and total mass of two particles, respectively. For the δ shell potential, this integral equation is soluble by using the partial wave analysis. The method is a trivial extension of the calculation of Brueckner and Sawada[22]; we will therefore only quote the result:

$$t_{pp'l}(\omega_k) = \begin{cases} -\dfrac{2\pi\lambda a^2}{\mu} \dfrac{j_l(pa) j_l(p'a)}{1 - i\lambda a^2 k j_l(ka) h_l^{(1)}(ka)}, & \text{for } \operatorname{Im}\omega_k > 0 \quad (5.5a) \\ -\dfrac{2\pi\lambda a^2}{\mu} \dfrac{j_l(pa) j_l(p'a)}{1 + i\lambda a^2 k j_l(ka) h_l^{(2)}(ka)}, & \text{for } \operatorname{Im}\omega_k < 0 \quad (5.5b) \end{cases}$$

where

$$t_{\mathbf{pp}'}(\omega_k) = \frac{1}{(2\pi)^3 \Omega} \sum_{l=0}^{\infty} (2l+1) t_{pp'l}(\omega_k) P_l(\cos \widehat{\mathbf{pp}'}) \tag{5.6}$$

and j_l, $h_l^{(1)}$, and $h_l^{(2)}$ are the spherical Bessel functions, and P_l is the Legendre function, and

$$\omega_k = \frac{k^2}{2\mu} \tag{5.7}$$

From the definition of the self-energy part $G_\alpha(z)$ in (A.10), $\operatorname{Im}[N t_{\mathbf{pp}}(E \pm i0)]$ corresponds to $\pm \nu_2 J_\alpha$ in Eq. (3.7), and the condition (3.11) is expressed in the second-order approximation on c by

$$\left| \frac{n_p \operatorname{Im}\left[N t_{\mathbf{pp}}(\omega_p + i0) \right]}{p^2/2\mu + P^2/2M + \operatorname{Re}\left[N t_{\mathbf{pp}}(\omega_p + i0) \right]} \right| \ll 1 \tag{5.8}$$

where

$$n_p = 1 + \left\{ \frac{\partial}{\partial E} \operatorname{Re}\left[N t_{\mathbf{pp}}(E) \right] \right\}_{E = \omega_p + i0} \tag{5.9}$$

Now we evaluate the left-hand side of Eq. (5.8) for each case of low-energy scattering, resonance scattering, and the high-energy scattering for the δ shell potential and the hard-sphere potential.

5.1. Low-Energy Scattering

Because the s wave scattering is dominant for low-energy scattering, $pa \ll 1$, we obtain from Eq. (5.5a)

$$Nt_{\mathrm{pp}}(\omega p + i0) \approx -\frac{2\pi\lambda a^2 c}{\mu} \frac{(1-\lambda a) + i\lambda a \cdot pa}{(1-\lambda a)^2} \tag{5.10}$$

and

$$\left[\frac{\partial}{\partial E} Nt_{\mathrm{pp}}(E)\right]_{E=\omega_p+i0} \approx -2\pi i \frac{a^3 c}{pa} \frac{(\lambda a)^2}{(1-\lambda a)^2} \tag{5.11}$$

Thus we may estimate the left-hand side of Eq. (5.8) for the low-energy case, $pa \ll 1$ and $Pa \ll 1$, by

$$\left| \frac{n_p \mathrm{Im}\left[Nt_{\mathrm{pp}}(\omega_p + i0)\right]}{(p^2/2\mu) + (P^2/2M) + \mathrm{Re}\left[Nt_{\mathrm{pp}}(\omega_p + i0)\right]} \right| \approx \left| \frac{\lambda a}{1-\lambda a} \right| pa \tag{5.12}$$

Therefore, the condition (3.11) is satisfied except for the case $\lambda a \approx 1$.

To verify the validity of the kinetic equation, we now show the regularity of the kinetic s.operators at $l = +i0$. The imaginary part of Eq. (5.10) which corresponds to $-\gamma_\alpha$ in Eq. (3.9) also corresponds to the loss part of the collision s.operator [see (A.16)] at $l = +i0$. Thus this part is regular except for $\lambda a \approx 1$. The regularity of the remaining parts of the kinetic s.operators other than $\Delta_E(l)$ can be shown in a similar discussion. The argument for $\Delta_E(l)$ is more delicate; in order to ensure the regularity, γ_α should be a finite value, even though γ_α is very small to ensure that $\Delta_E(+i0)$ has a sharp peak at $E = E_\alpha$.

We further notice that the magnitude of the real part of the derivative of the T matrix, $\mathrm{Re}[\partial Nt_{\mathrm{pp}}/\partial E]$, in Eq. (5.11) gives the magnitude of the memory effect on time t, i.e., $\partial_l \chi_E$. Indeed, from the definition of the argument E and l, the real part, $\mathrm{Re}[\partial Nt_{\mathrm{pp}}/\partial E]$, just corresponds to the derivative of the loss part of the collision s.operator on l, $\partial_l \mathcal{G}_E(l)$, and that the structure of the gain part is the same with the loss part [see (A.15), (A.16), and (A.18)]. Therefore, we see from Eq. (5.11) that we can neglect the last term in Eq. (4.10) for low-energy scattering. (The detailed calculation of $\mathrm{Re}[\partial Nt_{\mathrm{pp}}/\partial E]$ shows that it is a higher-order term than pa.) By taking the limit $\lambda a \to \infty$ in Eqs. (5.10)–(5.12), we see that similar statements hold for the case of the hard-sphere potential.

5.2. Resonance Scattering

We next consider the scattering having the energy near the s wave resonance under the assumption of strong coupling interaction,

$$\lambda a \gg 1 \tag{5.13}$$

The resonance energies are in general given by the zeros of the denominator of Eq. (5.5) near the real axis of ka. Equation (5.5) shows that because of the condition (5.13) these zeros must lie near the solution of $j_l(ka) = 0$. Therefore, from the well-known property of zeros of the Bessel function, we see that near the s wave resonance energy there is no resonance contribution of the other partial waves, $l \neq 0$, except for the case of $ka \approx 0$ which was already discussed in the previous paragraph. Thus, we can neglect the contributions of the other partial waves and obtain the explicit form of the T matrix near the s wave resonance[21]:

$$Nt_{\mathrm{pp}}(\omega_p + i0) \approx -\frac{2\pi ac}{\mu}\,\frac{\eta_n}{pa}\,\frac{-(pa-\xi_n)+i\eta_n}{(pa-\xi_n)^2+\eta_n^2} \tag{5.14}$$

and

$$\left[\frac{\partial}{\partial E}Nt_{\mathrm{pp}}(E)\right]_{E=\omega_p+i0} \approx -\frac{2\pi a^3 c\eta_n}{(pa)^3}\,\frac{\xi_n\left[(pa-\xi_n)^2+\eta_n^2\right]-2pa\eta_n^2+i\eta_n\left[\xi_n^2-(pa)^2+\eta_n^2\right]}{\left[(pa-\xi_n)^2+\eta_n^2\right]^2} \tag{5.15}$$

where the resonance occurs at $pa = \xi_n$:

$$\xi_n = n\pi\left(1+\frac{1}{\lambda a}\right), \qquad n = \pm 1, \pm 2, \ldots \tag{5.16}$$

and the width of the resonance η_n is given by

$$\eta_n = \left(\frac{n\pi}{\lambda a}\right)^2 \ll 1 \tag{5.17}$$

At the resonance $pa = \xi_n$, we have

$$\left|\frac{n_p \mathrm{Im}\left[Nt_{\mathrm{pp}}(\omega_p+i0)\right]}{(p^2/2\mu)+(P^2/2M)+\mathrm{Re}\left[Nt_{\mathrm{pp}}(\omega_p+i0)\right]}\right| \leqslant \left|\frac{n_p \mathrm{Im}\left[Nt_{\mathrm{pp}}(\omega_p+i0)\right]}{(p^2/2\mu)}\right| \approx 2\left(1+\frac{2\pi a^3 c}{\xi_n^2\eta_n}\right)\frac{2\pi a^3 c}{\xi_n^2\eta_n}\cdot\frac{\eta_n}{\xi_n} \tag{5.18}$$

The imaginary part of Eq. (5.14) shows that for small but finite η_n the width

of $\Delta_E(+i0)$, γ_α, in Eq. (3.9), is finite (positive), and thus $\Delta_E(l)$ is regular at $l = +i0$.

The right-hand side of the inequality (5.18) can be evaluated as follows: In order to verify the δ function approximation of $\Delta_E(+i0)$, Eqs. (5.14) and (5.15) should be slowly varying functions as compared with $\Delta_E(+i0)$. This implies that the energy width of the resonance $\gamma_n \equiv (\xi_n \eta_n)/(a^2\mu)$ which corresponds to the momentum width η_n should be much larger than the width γ_α in Eq. (3.9), i.e., $\gamma_\alpha/\gamma_n \ll 1$, or equivalently,

$$\left(1 + \frac{2\pi a^3 c}{\xi_n^2 \eta_n}\right) \frac{2\pi a^3 c}{\xi_n^2 \eta_n} \ll 1 \tag{5.19}$$

Combining this inequality with Eq. (5.18), we see that the condition (5.8) holds. We notice that the inequality (5.19) also ensures the regularity of the kinetic s.operators at $l = +i0$. We further notice that Eq. (5.19) gives an upper limit of λa; combining it with the inequality (5.13), we have

$$1 \ll \lambda a \ll \frac{(n\pi)^2}{(2\pi a^3 c)^{1/2}} \tag{5.20}$$

The left inequality is needed to ensure the resonance scattering, the right inequality to verify the quantum kinetic equation (4.10).

As is well known, the inverse of the energy width of the resonance, $\tau_d \equiv (2\gamma_n)^{-1}$, is the time delay of the resonance scattering.[19,21] On the other hand, we will show in the last section that the inverse of γ_α, i.e., $\tau_r \equiv (2\gamma_\alpha)^{-1}$, is just the relaxation time of the dissipative system. Therefore, the inequalities (5.18) and (5.19) show that if the scale of the time delay and the relaxation time is sufficiently separated such that

$$\tau_r \gg \tau_d \tag{5.21}$$

then the quantum kinetic equation (4.10) is valid, even though the resonance scattering occurs. Furthermore, we can see from Eq. (5.15) that the memory effect on time t, i.e., the real part of Eq. (5.15) at $pa = \xi_n$, is the same order of the ratio τ_d/τ_r. Thus, the last term in Eq. (4.10) is negligibly small.

It may be interesting to compare the relaxation time τ_r with the mean free time τ_m. Because $\lambda a \gg 1$, τ_m may be approximated by hard sphere's one, i.e.,

$$\tau_m \approx \left(\frac{1}{4\sqrt{2}\,\pi a^2 c}\right) \Big/ \left(\frac{\xi_n}{\mu a}\right) = \frac{\sqrt{2}}{4\xi_n^2}\left(1 + \frac{2\pi a^3 c}{\xi_n^2 \eta_n}\right)\tau_r \tag{5.22}$$

It shows that the relaxation time is several times larger than the mean free time.

5.3. High-Energy Scattering

The final example is the high-energy scattering, $pa \gg 1$, for the hard-sphere potential. The T matrix for this system is given by dropping unity in the denominator of Eq. (5.5).[21] In the high-energy limit, i.e., the classical limit, the impact parameter, l/p, has an upper limit, $l_{\max}/p \approx a$. By using asymptotic forms of the spherical Bessel functions, we obtain

$$Nt_{\mathbf{pp}}(\omega_p + i0) \approx -i\frac{\pi ac}{\mu pa}\sum_{l=0}^{pa}(2l+1)(e^{-i[2pa-(l+1)\pi]}+1)$$
$$\approx \frac{\pi ac}{\mu}\{\sin(\pi-2)pa - i[pa + 2 + \cos(\pi-2)pa]\} \quad (5.23)$$

and

$$\left[\frac{\partial}{\partial E}Nt_{\mathbf{pp}}(E)\right]_{E=\omega_p+i0} \approx -\frac{2\pi a^3 c}{(pa)^2}\left\{-(pa)^2 - pa + \frac{1}{2}\sin(\pi-2)pa\right.$$
$$\left. - \frac{i}{2}[pa + 2 + \cos(\pi-2)pa]\right\} \quad (5.24)$$

Thus, we have

$$\left|\frac{n_p \mathrm{Im}[Nt_{\mathbf{pp}}(\omega_p+i0)]}{(p^2/2\mu)+(P^2/2M)+\mathrm{Re}[Nt_{\mathbf{pp}}(\omega_p+i0)]}\right|$$
$$\leqslant \left|\frac{N_r \mathrm{Im}[Nt_{\mathbf{pp}}(\omega_p+i0)]}{(p^2/2\mu)}\right| \approx (1+2\pi a^3 c)\frac{2\pi a^3 c}{pa} \ll 1 \quad (5.25)$$

In a similar discussion as above, we can see that all assumptions imposed in Section 3 are satisfied in this system.

An interesting difference from the previous examples is that the memory effect on time t cannot be neglected for this system. Indeed, Eq. (5.24) shows that its real part is proportional to $2\pi a^3 c$ without other small factors. Thus, in the second approximation of the density expansion, we cannot neglect it.

In this case we can also calculate the ratio between the relaxation time τ_r and the mean free time τ_m explicitly:

$$\frac{\tau_m}{\tau_r} \approx \frac{1}{2\sqrt{2}} \quad (5.26)$$

The above three examples show that the quantum kinetic equation which describes the effect of two- and three-body scattering may exist for a fairly wide region of the energy if the time scales are sufficiently separated.

Finally, we mention the convergence of the series expansion of $P\Gamma_E P$ and $P\tilde{\Gamma}_E P$ in Eq. (3.3) obtained by the iterating procedure of (A.23). In

order to establish the quantum kinetic equation (4.10), we should, of course, prove its convergence. Because of its complex structure, however, it is still an open problem. We can only say here that all the examples given in this section suggest the convergence, since they show that the memory effect, $\partial_l \chi_E$, which is the second approximation of the series of $P\Gamma_E P$, is always very small compared with unity.

6. EPILOGUE: TIME EVOLUTION OF $\rho_s(t)$ ON s

In closing we would like to mention the physical role of the two-time dyad, $\rho_s(t)$, as a function of s. Equation (2.9) shows that its evolution on s is governed by the energy superoperator, $\hat{H}$, while the evolution on t is governed by the Liouvillian $\tilde{H}$. Further, Eq. (2.7) shows that the evolution on s relates to the surviving amplitude. This fact suggests that the dissipativity of the system may be understood by investigating the evolution of $\rho_s(t)$ on s for a fixed t. This inference is indeed justified if we calculate the contribution through the width γ_α in $\Delta_E(+i0)$ more precisely, which we have neglected in the δ-function approximation Eq. (3.12). Combining Eqs. (3.1) and (3.2) with the first terms of Eq. (3.5) and Eq. (3.9) including a finite width γ_α, we have the dominant term of the solution for $(\alpha;\alpha|\rho_s^{\mathrm{as}}(t))$

$$\begin{aligned}(\alpha;\alpha|\rho_s^{\mathrm{as}}(t)) &\approx \frac{n_\alpha}{\pi}\int_{+\infty}^{-\infty} dE\, e^{-iEs}\frac{\gamma_\alpha}{(E-E_\alpha)^2+\gamma_\alpha^2} \\ &\quad\times\left(\alpha;\alpha|(1+\partial_l\tilde{\chi}_{E_\alpha})e^{-iP\tilde{\Gamma}_{E_\alpha}Pt}(P+P\mathfrak{D}_{E_\alpha}Q)|\rho(0)\right) \\ &= n_\alpha e^{-iE_\alpha s-\gamma_\alpha|s|}\left(\alpha;\alpha|(1+\partial_l\tilde{\chi}_{E_\alpha})e^{-iP\tilde{\Gamma}_E\alpha Pt}(P+P\mathfrak{D}_{E_\alpha}Q)|\rho(0)\right)\end{aligned} \tag{6.1}$$

For $s=0$ we obtain the same solution which we utilized to obtain the quantum kinetic equation.

Similarly, from the second term of Eq. (3.9) and from Eq. (3.13), we can calculate the additional term to Eq. (6.1) for the following two cases: (i) for $s=0$, the additional term, $\delta(\alpha;\alpha|\rho_s^{\mathrm{as}}(t))$ is zero, and (ii) for $s\neq 0$,

$$\begin{aligned}\delta(\alpha;\alpha|\rho_s^{\mathrm{as}}(t)) &\approx e^{-iE_\alpha s\mp\gamma_\alpha s}\sum_\beta\left\{\pm in_\alpha\gamma_\alpha\frac{J'_\alpha(E_\alpha)}{J_\alpha(E_\alpha)}(\alpha;\alpha|(1+\partial_l\tilde{\chi}_{E_\alpha})|\beta;\beta)\right. \\ &\quad\left.-\gamma_\alpha s\left[n_\alpha\gamma_\alpha\frac{J'_\alpha(E_\alpha)}{J_\alpha(E_\alpha)}\pm 1\right]\left[\frac{(\alpha;\alpha|\mathfrak{W}_{E_\alpha}\Delta_{E_\alpha}|\beta;\beta)}{2iJ_\alpha(E_\alpha)}-\delta_{\alpha,\beta}\right]\right\} \\ &\quad\times\left(\beta;\beta|e^{-iP\tilde{\Gamma}_E\alpha Pt}(P+P\mathfrak{D}_{E_\alpha}Q)|\rho(0)\right)\end{aligned} \tag{6.2}$$

where upper (lower) signs are taken together for $s > 0$ ($s < 0$). For sufficiently small γ_α and sufficiently short time on s such that $\gamma_\alpha |s| \ll 1$, we can neglect the additional term, as expected in Section 3. On the other hand, when $|s|$ increases and $\gamma_\alpha |s|$ is the order of unity, the additional term cannot be neglected as compared with Eq. (6.1).[4] However, whole contributions in Eqs. (6.1) and (6.2) tend to zero due to the damping factor. Therefore, we may conclude

$$\{\langle \psi(t)|\psi(t+s)\rangle\}_{\text{av}} = \sum_\alpha (\alpha;\alpha|\rho_s(t)) \xrightarrow[|s|\to\infty]{} 0 \tag{6.3}$$

for sufficiently large t. This is the statement about the dissipativity of the system which we expected.

It should be noted that dissipativity is ensured by $J_\alpha(E_\alpha) \neq 0$, i.e., the finite width of $\Delta_E(+i0)$, and this is just the same condition as Van Hove's.[2] From this result we see that it is natural to define the relaxation time as $\tau_r = (2\gamma_\alpha)^{-1}$, because the surviving probability is the square of the absolute value of its amplitude.

Finally, we would like to mention to the reduced single particle function of $\text{tr}\,\rho_s(t)$; it is essentially the two-time Green function.[13] Our formalism shows that its evolution on the relative time is governed by the energy s.operator $\hat{H}$, rather than by the Liouvillian $\tilde{H}$ [see Eqs. (2.9) and (2.11)]. It may be interesting, therefore, to reformulate the theory of the two-time correlation function from this point of view. This will be done elsewhere.

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APPENDIX A: SOME DEFINITIONS AND RESULTS[5]

Perturbed and unperturbed resolvent operators in the ordinary Hilbert space $\mathfrak{K}$ are defined by $R(z) = (H_0 + V - z)^{-1}$ and $R_0(z) = (H_0 - z)^{-1}$, respectively. They are related through the resolvent equation

$$R(z) = R_0(z) - R_0(z)VR(z) \tag{A1}$$

[4] We notice that the master equation (3.15) still varied even though the additional term cannot be neglected.

[5] The details of calculations in this appendix are given in Refs. 12 and 14.

and its iterated solution is given by

$$R(z) = R_0(z) \sum_{k=0}^{\infty} \left[-VR_0(z) \right]^k \tag{A2}$$

In the superspace $\mathbb{S}$, the projection s.operators P, which projects out the vacuum component of the correlation, and Q, which projects out the correlation component, are introduced such that

$$P = \sum_{\alpha} |\alpha;\alpha)(\alpha;\alpha|, \qquad Q = 1 - P \tag{A3}$$

where Greek letters are used to express the labels of the eigenstate of the unperturbed Hamiltonian, $H_0|\alpha\rangle = \epsilon_\alpha|\alpha\rangle$, and the s.vector $|\alpha;\beta)$ is defined through a dyad operator in $\mathcal{K}$ by $|\alpha;\beta) \equiv ||\alpha\rangle\langle\beta|)$. They satisfy the relations

$$P^2 = P^+ = P, \qquad Q^2 = Q^+ = Q, \qquad PQ = QP = 0 \tag{A4}$$

Introducing the kinetic s.operators which are defined in Eqs. (A6)–(A12), the product of two resolvent s.operators, $R^> R^<$, in $\mathbb{S}$ is decomposed in the following form:

$$R^>(E + \tfrac{1}{2}l)R^<(E - \tfrac{1}{2}l) = \left[P + Q\mathcal{C}_E(l)P \right] R^>(E + \tfrac{1}{2}l)R^<(E - \tfrac{1}{2}l) \times \left[P + P\mathcal{D}_E(l)Q \right] + Q\mathcal{G}_E(l)\,Q \tag{A5}$$

Here, the creation-of-correlation (creation) s.operator is defined through the definitions (A9)–(A12) by

$$Q\mathcal{C}_E(l)P = Q\left[D^>(z)N^>(z) + D^<(z')N^<(z') + D^>(z)D^<(z')\{N^>(z)N^<(z')\}_{\text{t.ti}} \right] P \tag{A6}$$

where $z = E + \frac{1}{2}l$, $z' = E - \frac{1}{2}l$. The destruction-of-correlation (destruction) s.operator is defined by

$$P\mathcal{D}_E(l)\,Q = P\left[N^>(z)D^>(z) + N^<(z')D^<(z') + \{N^>(z)N^<(z')\}_{\text{t.ti}} D^>(z)D^<(z') \right] Q \tag{A7}$$

The propagation-of-correlation s.operator is defined by

$$Q\mathcal{G}_E(l)\,Q = QD^>(z)D^<(z')\left[1 + N^>(z)D^>(z) + N^<(z')D^<(z') + \{N^>(z)N^<(z')\}_{\text{t.ti}} D^>(z)D^<(z') \right] Q \tag{A8}$$

In these definitions the renormalized propagator is defined by

$$\langle\alpha|D(z)|\beta\rangle = D_\alpha(z)\delta_{\alpha,\beta} = \langle\alpha|\sum_{i=0}^{\infty} R_0(z)\big[G(z)R_0(z)\big]^i|\alpha\rangle\delta_{\alpha,\beta}$$
$$= \langle\alpha|\frac{1}{H_0 - z - G(z)}|\alpha\rangle\delta_{\alpha,\beta} \tag{A9}$$

and the self-energy part is defined by

$$\langle\alpha|G(z)|\beta\rangle = G_\alpha(z)\delta_{\alpha,\beta} = \langle\alpha|\left\{\sum_{i=0}^{\infty} -V\big[-R_0(z)V\big]^i\right\}_{\text{o.ti}}|\alpha\rangle\delta_{\alpha,\beta} \tag{A10}$$

and

$$\langle\alpha|N(z)|\beta\rangle = N_{\alpha\beta}(z)(1-\delta_{\alpha,\beta})$$
$$= \langle\alpha|\left\{\sum_{i=0}^{\infty} -V\big[-R_0(z)V\big]^i\right\}_{\text{o.ti}}|\beta\rangle(1-\delta_{\alpha,\beta}) \tag{A11}$$

and

$$(\alpha;\,\beta|\{N^>(z)N^<(z')\}_{\text{t.ti}}|\gamma;\delta)$$
$$= \langle\alpha|\left\{\sum_{i=0}^{\infty} -V\big[-R_0(z)V\big]^i|\gamma\rangle\langle\delta|\sum_{j=0}^{\infty} -V\big[-R_0(z')V\big]^j\right\}_{\text{t.ti}}|\beta\rangle$$
$$\times(1-\delta_{\alpha,\gamma})(1-\delta_{\beta,\delta}) \tag{A12}$$

Here the subscript o.ti stands for "one-side topologically irreducible," which implies that the intermediate states should never be identical to the initial and final states, $|\alpha\rangle$ and $|\beta\rangle$, while t.ti stands for "two-side topologically irreducible," which implies that the intermediate states on opposite sides of the operator $|\gamma\rangle\langle\delta|$ should never be identical to each other or to $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, and $|\delta\rangle$.[23]

The diagonal part in Eq. (A.5), i.e., P–P component, is expressed by

$$PR^>(E+\tfrac{1}{2}l)R^<(E-\tfrac{1}{2}l)P = \sum_{n=0}^{\infty}\left(\frac{1}{l}\right)^{n+1}\big[\chi_E(l)\big]^n\Delta_E(l)$$
$$= \Delta_E(l)\sum_{n=0}^{\infty}\left(\frac{1}{l}\right)^{n+1}\big[\tilde{\chi}_E(l)\big]^n \tag{A13}$$

where

$$\Delta_E(l) = P\big[D^>(z) - D^<(z')\big]P \tag{A14}$$
$$\mathcal{G}_E(l) = P\big[G^>(z) - G^<(z')\big]P \tag{A15}$$
$$\chi_E(l) = \Delta_E(l)\mathcal{W}_E(l) - \mathcal{G}_E(l) \tag{A16a}$$
$$\tilde{\chi}_E(l) = \mathcal{W}_E(l)\Delta_E(l) - \mathcal{G}_E(l) \tag{A16b}$$
$$\mathcal{W}_E(l) = P\{N^>(z)N^<(z')\}_{\text{t.ti}}P \tag{A17}$$

The kinetic s.operators χ and $\tilde{\chi}$ are called "collision s.operators." We distinguish the first terms of the collision s.operators, $\Delta\mathfrak{W}$ and $\mathfrak{W}\Delta$, as the "gain part" and the second term, $\mathcal{G}$, the "loss part." The loss part can be expressed in a similar form of the gain part:

$$(\alpha;\alpha|\mathcal{G}_E(l)|\alpha;\alpha) = \sum_\beta (\alpha;\alpha|\mathfrak{W}_E(l)\Delta_E(l)|\beta;\beta)$$
$$= \sum_\beta (\beta;\beta|\Delta_E(l)\mathfrak{W}_E(l)|\alpha;\alpha) \tag{A18}$$

The collision s.operators satisfy the relation

$$\chi_E(l)\Delta_E(l) = \Delta_E(l)\tilde{\chi}_E(l) \tag{A19}$$

In the decomposition (A5) with (A13), all singularities at $l = 0$ are seen explicitly. The elucidation of the singularities has previously been achieved with the aid of the diagrammatic technique in Ref. 14. To complete our comments we will here rederive Eq. (A13) by purely algebraic methods in Appendix B.

The asymptotic time evolution s.operator, $\Sigma_E(t)$, is defined through the contribution of the poles at $l = 0$ of the analytically continued function of $R^>(E + \frac{1}{2}l)R^<(E - \frac{1}{2}l)$ from the upper half-plane of l for $t > 0$. Assuming that the poles at $l = +i0$, which are seen explicitly in Eq. (A13), are isolated from the singularities of the analytically continued function of the kinetic s.operators into the lower half-plane, we obtain

$$\Sigma_E(t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_0''} dl\, e^{-ilt} R^>(E + \tfrac{1}{2}l)R^<(E - \tfrac{1}{2}l)$$
$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{n!}(-it + \partial_l)\left[P + Q\mathcal{C}_E(l)P\right]$$
$$\times \left\{\left[\chi_E(l)\right]^n\Delta_E(l) \text{ or } \Delta_E(l)\left[\tilde{\chi}_E(l)\right]^n\right\}\left[P + P\mathcal{D}_E(l)Q\right]_{l=+i0} \tag{A20}$$

where $\partial_l \equiv \partial/\partial l$ and the contour Γ_0'' encircles the poles at $l = +i0$ counterclockwise excluding all other singularities in the continued plane. We assume the convergence of the left-hand side of (A.20). Then we can show that $P\Sigma_E(t)P$ obeys the following asymptotic master equations[14]:

$$i\frac{\partial}{\partial t}P\Sigma_E(t)P = 2\pi\int_0^\infty dt'\,\chi'_E(t')P\Sigma_E(t-t')P \tag{A21a}$$

$$i\frac{\partial}{\partial t}P\Sigma_E(t)P = 2\pi\int_0^\infty dt'\,P\Sigma_E(t-t')P\tilde{\chi}'_E(t') \tag{A21b}$$

where

$$\chi'_E(t) = \left(\frac{1}{2\pi i}\right)^2\int_{\Gamma''} dl\, e^{-ilt}\chi_E(l) \tag{A22a}$$

$$\tilde{\chi}'_E(t) = \left(\frac{1}{2\pi i}\right)^2\int_{\Gamma''} dl\, e^{-ilt}\tilde{\chi}_E(l) \tag{A22b}$$

Introducing s.operators $P\Gamma_E P$ and $P\tilde{\Gamma}_E P$ as the solution of the following equations,

$$P\Gamma_E P = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_l^n \chi_E(l)|_{l=+i0} (P\Gamma_E P)^n \tag{A23a}$$

$$P\tilde{\Gamma}_E P = \sum_{n=0}^{\infty} \frac{1}{n!} (P\tilde{\Gamma}_E P)^n \partial_l^n \tilde{\chi}_E(l)|_{l=+i0} \tag{A23b}$$

we can rewrite the equations (A21) into the following forms:

$$i\frac{\partial}{\partial t} P\Sigma_E(t)P = P\Gamma_E P\Sigma_E(t)P \tag{A24a}$$

$$i\frac{\partial}{\partial t} P\Sigma_E(t)P = P\Sigma_E(t)P\tilde{\Gamma}_E P \tag{A24b}$$

Furthermore, the full component of $\Sigma_E(t)$ satisfies the relation

$$\Sigma_E(t) = [P + Q\mathbf{C}_E P]\Sigma_E(t)[P + P\mathbf{D}_E Q] \tag{A25}$$

where $Q\mathbf{C}_E P$ and $P\mathbf{D}_E Q$ are defined by

$$Q\mathbf{C}_E P = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_l^n Q\mathcal{C}_E(l)P|_{l=+i0} (P\Gamma_E P)^n \tag{A26a}$$

$$P\mathbf{D}_E Q = \sum_{n=0}^{\infty} \frac{1}{n!} (P\tilde{\Gamma}_E P)^n \partial_l^n P\mathcal{D}_E(l)Q|_{l=+i0} \tag{A26b}$$

APPENDIX B: DERIVATION OF EQ. (A13)

Let us first consider a term in the perturbation series of $PR^{>}(z)R^{<}(z)P$ which consists of $k-m$ left fragments $G^{>}$ and m right fragments $G^{<}$, i.e.,

$$\begin{aligned}
&(\alpha;\alpha|\left[\frac{1}{H_0^{>}-z}G^{>}(z)\right]^{k-m}\frac{1}{H_0^{>}-z}\left[\frac{1}{H_0^{<}-z'}G^{<}(z')\right]^{m}\frac{1}{H_0^{<}-z'}|\alpha;\alpha) \\
&\quad=\left(\frac{1}{\epsilon_\alpha - z}\right)^{k-m+1}\left(\frac{1}{\epsilon_\alpha - z'}\right)^{m+1} G_\alpha^{k-m}G_\alpha'^{m} \\
&\quad=\sum_{j=0}^{k-m}\binom{m+j}{j}\left(\frac{1}{z-z'}\right)^{m+j+1}(-G_\alpha)^j G_\alpha'^{m}\left[\frac{1}{\epsilon_\alpha - z - G_\alpha}\right]_{(k-m-j)} \\
&\qquad+\sum_{i=0}^{m}\binom{k-m+i}{k-m}\left(\frac{1}{z-z'}\right)^{k-m+i-1} \\
&\qquad\times(-G_\alpha)^{k-m}G_\alpha'^{i}\cdot(-1)\left[\frac{1}{\epsilon_\alpha - z' - G_\alpha'}\right]_{(m-i)}
\end{aligned} \tag{B1}$$

where we have used the abbreviations $G_\alpha \equiv G_\alpha(z)$, $G'_\alpha \equiv G_\alpha(z')$, and

$$\left[\frac{1}{\epsilon_\alpha - z - G_\alpha}\right]_{(n)} \equiv \left(\frac{1}{\epsilon_\alpha - z}\right)^{n+1} G_\alpha^n \tag{B2}$$

and made use of the following identity:

$$\begin{aligned}
&\left(\frac{1}{\epsilon - z}\right)^{n+1}\left(\frac{1}{\epsilon - z'}\right)^{m+1} \\
&\quad = \sum_{j=0}^{n} (-1)^j \binom{m+j}{m} \left(\frac{1}{z - z'}\right)^{m+j+1} \left(\frac{1}{\epsilon - z}\right)^{n-j+1} \\
&\qquad + \sum_{i=0}^{m} (-1)^i \binom{n+i}{n} \left(\frac{1}{z' - z}\right)^{n+i+1} \\
&\qquad \times \left(\frac{1}{\epsilon - z'}\right)^{n-i+1}
\end{aligned} \tag{B3}$$

This identity is proved as follows:

$$\begin{aligned}
\left(\frac{1}{\epsilon - z}\right)^{n+1}\left(\frac{1}{\epsilon - z'}\right)^{m+1} &= \frac{1}{n!\,m!}\left(\frac{\partial}{\partial z}\right)^n \left(\frac{\partial}{\partial z'}\right)^m \frac{1}{\epsilon - z}\,\frac{1}{\epsilon - z'} \\
&= \frac{1}{n!\,m!}\left(\frac{\partial}{\partial z}\right)^n \left(\frac{\partial}{\partial z'}\right)^m \frac{1}{z - z'}\left(\frac{1}{\epsilon - z} - \frac{1}{\epsilon - z'}\right)
\end{aligned} \tag{B4}$$

We first calculate the first term of Eq. (B4):

$$\begin{aligned}
&\frac{1}{n!\,m!}\left(\frac{\partial}{\partial z}\right)^n \left(\frac{\partial}{\partial z'}\right)^m \frac{1}{z - z'}\,\frac{1}{\epsilon - z} \\
&\quad = \frac{1}{n!}\left(\frac{\partial}{\partial z}\right)^n \left(\frac{1}{z - z'}\right)^{m+1} \frac{1}{\epsilon - z} \\
&\quad = \sum_{j=0}^{n} \frac{1}{j!\,(n-j)!}\left[\left(\frac{\partial}{\partial z}\right)^j \left(\frac{1}{z - z'}\right)^{m+1}\right]\left[\left(\frac{\partial}{\partial z}\right)^{n-j} \frac{1}{\epsilon - z}\right] \\
&\quad = \sum_{j=0}^{n} (-1)^j \frac{(m+j)!}{j!\,m!}\left(\frac{1}{z - z'}\right)^{m+j+1}\left(\frac{1}{\epsilon - z}\right)^{n-j+1}
\end{aligned} \tag{B5}$$

This is just the first term in Eq. (B3). The second term is obtained only by changing the variables $z \rightleftarrows z'$ and $n \rightleftarrows m$. Then the desired result is obtained.

In Eq. (B1) we sum on m from 0 to k and sum on k from 0 to ∞:

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k}(B\cdot 1)$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{z-z'}\right)^{k+1}\binom{k+j}{j}(-G_\alpha)^{k-j}G_\alpha'^{j}\sum_{n=0}^{\infty}\left[\frac{1}{\epsilon_\alpha - z - G_\alpha}\right]_{(n)}$$

$$- \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{z-z'}\right)^{k+1}\binom{k+j}{j}(-G_\alpha)^{j}G_\alpha'^{k-j}\sum_{n=0}^{\infty}\left[\frac{1}{\epsilon_\alpha - z' - G_\alpha'}\right]_{(n)}$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{z-z'}\right)^{k+1}(-G_\alpha + G_\alpha')^{k}\left(\frac{1}{\epsilon_\alpha - z - G_\alpha} - \frac{1}{\epsilon_\alpha - z' - G_\alpha'}\right)$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{l}\right)^{k+1}(\alpha;\alpha|\left[-\mathcal{G}_E(l)\right]^{k}\Delta_E(l)|\alpha;\alpha) \tag{B6}$$

where $l = z - z'$ and $E = (z + z')/2$. Therefore, the whole summation in the perturbation series $PR^{>}R^{<}P$ is given by

$$\begin{aligned}
PR^{>}R^{<}P = {} & \sum_{k=0}^{\infty}\left(\frac{1}{l}\right)^{k+1}P\left[-\mathcal{G}_E(l)\right]^{k}\Delta_E(l)P \\
& + \left\{\sum_{k=0}^{\infty}\left(\frac{1}{l}\right)^{k+1}P\left[-\mathcal{G}_E(l)\right]^{k}\Delta_E(l)P\right\} \\
& \times \mathcal{W}_E(l)\left\{\sum_{k'=0}^{\infty}\left(\frac{1}{l}\right)^{k'+1}P\left[-\mathcal{G}_E(l)\right]^{k'}\Delta_E(l)P\right\} \\
& + \left\{\sum_{k=0}^{\infty}\left(\frac{1}{l}\right)^{k+1}P\left[-\mathcal{G}_E(l)\right]^{k}\Delta_E(l)P\right\} \\
& \times \mathcal{W}_E(l)\left\{\sum_{k'=0}^{\infty}\left(\frac{1}{l}\right)^{k'+1}P\left[-\mathcal{G}_E(l)\right]^{k'}\Delta_E(l)P\right\} \\
& \times \mathcal{W}_E(l)\left\{\sum_{k''=0}^{\infty}\left(\frac{1}{l}\right)^{k''+1}P\left[-\mathcal{G}_E(l)\right]^{k''}\Delta_E(l)P\right\} + \cdots
\end{aligned} \tag{B7}$$

Rearranging Eq. (B7) according to the same order of l^{-1}, we obtain

$$\begin{aligned}
PR^{>}R^{<}P &= \sum_{n=0}^{\infty}\left(\frac{1}{l}\right)^{n+1}\left[\Delta_E(l)\mathcal{W}_E(l) - \mathcal{G}_E(l)\right]^{n}\Delta_E(l) \\
&= \sum_{n=0}^{\infty}\Delta_E(l)\left(\frac{1}{l}\right)^{n+1}\left[\mathcal{W}_E(l)\Delta_E(l) - \mathcal{G}_E(l)\right]^{n}
\end{aligned} \tag{B8}$$

This is just Eq. (A13).

We remark that the result (B8) can be obtained also by a simple calculation from the following equation[12]:

$$PR^{>}(E+\tfrac{1}{2}l)R^{<}(E-\tfrac{1}{2}l)P = PD^{>}(E+\tfrac{1}{2}l)D^{<}(E-\tfrac{1}{2}l)P + PD^{>}(E+\tfrac{1}{2}l)D^{<}(E-\tfrac{1}{2}l)P \times \mathfrak{V}_E(l)PR^{>}(E+\tfrac{1}{2}l)R^{<}(E-\tfrac{1}{2}l)P \tag{B9}$$

However, the advantage of the above somewhat lengthy calculation is that its procedure has a direct correspondence with the diagrammatic technique introduced in Ref. 14. Therefore, we can construct the master equation step by step through the perturbation series which is resummed by a characteristic parameter of the system, such as the density, or the strength of the interaction.

APPENDIX C: THE FACTORIZATION THEOREM

In this Appendix we will prove the factorization theorem for the two-time reduced momentum distribution function which appears in the quantum kinetic equation under the operation of the collision s.operator:

$$\phi^{(m)}(\mathbf{p}_1, \ldots, \mathbf{p}_m; s, t) = \prod_{i=1}^{m} \phi^{(1)}(\mathbf{p}_i; s, t) \tag{C1}$$

Here the two-time distribution functions are defined in a similar way to the ordinary distribution function but replace the density matrix by the two-time dyad. That is, the N-particle Wigner distribution function and its reduced distribution functions are defined by

$$F^N(\{\mathbf{x},\mathbf{p}\}^N; s, t) = \frac{1}{(2\pi)^{3N}} \sum_{\mathbf{p}^N} \exp\left(-i\sum_{i=1}^{n} \mathbf{k}_i\mathbf{x}_i\right)\left(\left\{\mathbf{p}+\frac{\mathbf{k}}{2}\right\}^N; \left\{\mathbf{p}-\frac{\mathbf{k}}{2}\right\}^N \middle| \rho_s(t)\right) \tag{C2}$$

$$f^{(m)}(\{\mathbf{x},\mathbf{p}\}^m; s, t) = \frac{N!}{(N-m)!} \int (d\mathbf{x})^{N-m} \int_{\mathbf{p}^{N-m}} F^N(\{\mathbf{x},\mathbf{p}\}^N; s, t) \tag{C3}$$

$$\phi^{(m)}(\mathbf{p}^m; s, t) = \int (d\mathbf{x})^N \int_{\mathbf{p}^{N-m}} F^N(\{\mathbf{x},\mathbf{p}\}^N; s, t) \tag{C4}$$

and the "regular" Fourier coefficient of the Wigner function is defined by

$$\rho_{\mathbf{k}^m}(\mathbf{p}^m \mid \mathbf{p}^{N-m}; s, t) = \Omega^{N+\nu_m}\left(\left\{\mathbf{p}+\frac{\mathbf{k}}{2}\right\}^m, \mathbf{p}^{N-m}; \left\{\mathbf{p}-\frac{\mathbf{k}}{2}\right\}^m, \mathbf{p}^{N-m} \middle| \rho_s(t)\right) \tag{C5}$$

where $\mathbf{p}^m \equiv \{\mathbf{p}_1, \ldots, \mathbf{p}_m\}$, $\{\mathbf{x}, \mathbf{p}\}^m \equiv \{\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{p}_1, \ldots, \mathbf{p}_m\}$, $\{\mathbf{p} \pm (\mathbf{k}/2)\}^m \equiv \{\mathbf{p}_1 \pm (\mathbf{k}_1/2), \ldots, \mathbf{p}_m \pm (\mathbf{k}_m/2)\}$, and $\sum \mathbf{p}^m \equiv \sum \mathbf{p}_1 \ldots \sum \mathbf{p}_m$, and the summation $\int_{\mathbf{p}} \equiv \Omega^{-1} \sum_{\mathbf{p}}$ becomes the integration on $\mathbf{p}$ in the thermodynamic limit (4.9). The integer ν_m is the number of independent nonzero relative momenta, $\mathbf{k}$, of the off-diagonal component of the two-time dyad.[3] The coefficient having all $\mathbf{k} = 0$ is just the two-time N-particle momentum distribution function, i.e., $\rho_0(\mathbf{p}^N; s, t) = \phi^{(N)}(\mathbf{p}^N; s, t)$.

We further introduce the two-time spatial correlation functions $g^{(m)}(1, \ldots, m; s, t) \equiv g^{(m)}(\{\mathbf{x}, \mathbf{p}\}^m; s, t)$ for $m \geqslant s$, which are defined by a similar decomposition to the well-known cluster expansion of the ordinary distribution functions.[5] First typical expansions are

$$\begin{aligned} f^{(2)}(1,2) &= f^{(1)}(1) f^{(1)}(2) + g^{(2)}(1,2) \\ f^{(3)}(1,2,3) &= f^{(1)}(1) f^{(1)}(2) f^{(1)}(3) + f^{(1)}(1) g^{(2)}(2,3) \\ &\quad + f^{(1)}(2) g^{(2)}(3,1) + f^{(1)}(3) g^{(2)}(1,2) + g^{(3)}(1,2,3) \end{aligned} \tag{C6}$$

All above distribution functions are reduced to the ordinary distribution functions when $s = 0$.

To prove the factorization theorem (C1), we impose the following two basic assumptions: (i) we consider the system in the thermodynamic limit, (4.9); (ii) at the initial time $s = 0$, $t = 0$, the spatial correlation between particles vanishes if the distance of two particles increases, i.e.,

$$g^{(m)}(1, \ldots, m) \to 0 \qquad \text{when} \qquad |\mathbf{x}_i - \mathbf{x}_j| \to \infty \qquad \text{for} \qquad s = 0, t = 0 \tag{C7}$$

where i and j are any pair of particles in $1, \ldots, m$. Under assumption (ii) together with the following assumption that the Fourier coefficient $\rho_{\mathbf{k}^r}$ is not a pathological function on $\mathbf{k}$ at $s = 0$, $t = 0$, we can show that the Fourier coefficient is factorized by the reduced coefficient at $s = 0$, $t = 0$ such that[24]

$$\rho_{\mathbf{k}^r}(\mathbf{p}^r | \mathbf{p}^{N-r}) = \rho^{(l)}_{\mathbf{k}^l}(\mathbf{p}^l) \rho^{(m)}_{\mathbf{k}'^m}(\mathbf{p}^m) \cdots \rho^{(n)}_{\mathbf{k}''^n}(\mathbf{p}^n) \prod_{i=r+1}^{N} \phi^{(1)}(\mathbf{p}_i) \tag{C8}$$

where $l + m + \cdots + n = r$, and $\sum_i^l \mathbf{k}_i \neq 0$ and $\sum_j^m \mathbf{k}'_j = 0, \ldots, \sum_k^n \mathbf{k}''_k = 0$, and the reduced coefficient is defined by

$$\rho^{(m)}_{\mathbf{k}^m}(\mathbf{p}^m; s, t) = \int_{\mathbf{p}^{N-m}} \rho_{\mathbf{k}^m}(\mathbf{p}^m | \mathbf{p}^{N-m}; s, t) \tag{C9}$$

We now prove the factorization theorem. The proof will be performed through the following three steps:

1. In the reduced generalized master equation to the one-particle function, we will express the m-particle momentum distribution function appearing in the right-hand side of the master equation, $\phi^{(m)}(\mathbf{p}_1, \ldots, \mathbf{p}_m;$

s', t'), in the form of the perturbation series. Then we will show that the perturbation series for $\phi^{(m)}$ consists of m disconnected components, each of which includes one of the particles in $1, \ldots, m$. Here we call components "disconnected" if the components in each term of the perturbation series are not connected through interactions.

2. Next we show that this disconnected series of $\phi^{(m)}$ can be expressed by the convolution integrable of m-disconnected components on variables E and l.

3. Finally, by the Fourier–Laplace transform of this convolution integral, we obtain the product expression (C1) of the m-particle distribution function.

Let us introduce a terminology of "fixed particle" which designates the particle 1 of which momentum is not summed up when we reduce the master equation to the quantum kinetic equation (4.10); the remaining particles are called unfixed particles. In the thermodynamic limit, we can neglect a term in the perturbation series which includes a particle suffering a recollision after disappearing into the vacuum s.state $|\alpha; \alpha)$, through the interaction.[14] Therefore, in order to show the disconnected property of $\phi^{(m)}$ in the reduced master equation, it is sufficient to show that in the collision s.operator in the master equation we can neglect the contribution from such terms that consist of only unfixed particles, or that consist of several disconnected components; the second type of this contribution appears when we consider the contribution of four or more particles scattering.

The first part of this assertion is proved as follows. For the collision s.operator which consists of only unfixed particles, we have the following expression:

$$\sum_{\alpha}{}' \sum_{\beta} (\alpha; \alpha|\chi_E(l)|\beta; \beta)(\beta; \beta| = \sum_{\alpha}{}' \Big[\sum_{\beta} (\alpha; \alpha|\Delta_E(l)\mathfrak{W}_E(l)|\beta; \beta)(\beta; \beta| - (\alpha; \alpha|\mathcal{G}_E(l)|\alpha; \alpha)(\alpha; \alpha| \Big] \quad \text{(C10)}$$

where the prime on the summation sign indicates that the summation is not taken for the momentum of the fixed particle. Using Eq. (A18) and interchanging the dummy variables α and β in the second term of Eq. (C10), we can see that (C10) vanishes because the collision s.operator does not include the fixed particle. Therefore, our statement for the disconnected property is proved if we neglect the contribution coming from more than three-body scattering in the collision s.operator.

For the case of more general scattering, including four or more particles, the situation is more complicated, because the collision s.operator has a contribution from the term which consists of disconnected components, one of which includes the fixed particle and others consisting of only

unfixed particles. In order to prove the disconnected property for this general case, let us consider a particular contribution of the order $(V^{>})^{i}(V^{<})^{j}$ to $\rho_0(|\mathbf{p}^N; s', t')$ in the perturbed series

$$\rho_0(|\mathbf{p}^N; s', t') = \mathcal{K}_{ij}(\mathbf{p}^N, \mathbf{0} \,|\, \mathbf{p}'^N, \mathbf{k}^m; s', t')\rho_{\mathbf{k}^m}(\mathbf{p}'^m \,|\, \mathbf{p}'^{N-m}; 0, 0) \tag{C11}$$

where the evolution superoperator $\mathcal{K}_{ij}$ is one particular term in the development of

$$\begin{aligned}\left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{+\infty} dE\, e^{-iEs'} \int_{\Gamma''} dl\, e^{-ilt'} (-1)^{i+j} \\ \times (\mathbf{p}^N : \mathbf{0} | \left[(H_0^{>} - E - \tfrac{1}{2}l)^{-1} V^{>} \right]^i (H_0^{>} - E - \tfrac{1}{2}l)^{-1} \\ \times \left[(H_0^{<} - E + \tfrac{1}{2}l)^{-1} V^{<} \right]^j (H_0^{<} - E + \tfrac{1}{2}l) | \mathbf{p}'^N : \mathbf{k}^m)\end{aligned} \tag{C12}$$

and we use the abbreviation $|\mathbf{p}^N : \mathbf{k}^m) \equiv |\{\mathbf{p} + \frac{1}{2}\mathbf{k}\}^m, \mathbf{p}^{N-m}; \{\mathbf{p} - \frac{1}{2}\mathbf{k}\}^m, \mathbf{p}^{N-m})$. Using the convolution theorem on E and l, we can rewrite Eq. (C12), for $i \geqslant 1$ or $j \geqslant 1$, in the form

$$\begin{aligned}\rho(|\mathbf{p}^N; s', t') = \sum_n \sum_{\mathbf{k}''^n} \sum_{\mathbf{p}''^N} \int_{-\infty}^{+\infty} ds' \int_0^{\infty} dt' \\ \times \mathcal{K}_{i-i', j-j'}(\mathbf{p}^N, \mathbf{0} \,|\, \mathbf{p}''^N, \mathbf{k}''^n; s' - s'', t' - t'') \\ \times \hat{\rho}_{\mathbf{k}''^n}(\mathbf{p}''^n \,|\, \mathbf{p}''^{N-n}; s'', t'')\end{aligned} \tag{C13}$$

and

$$\begin{aligned}\hat{\rho}_{\mathbf{k}''^n}(\mathbf{p}''^n \,|\, \mathbf{p}''^{N-n}; s'', t'') = \hat{\mathcal{K}}_{i'j'}(\mathbf{p}''^N, \mathbf{k}''^n \,|\, \mathbf{p}'^N, \mathbf{k}^m; s'', t'') \\ \times \rho_{\mathbf{k}^m}(\mathbf{p}'^m \,|\, \mathbf{p}'^{N-m}; 0, 0)\end{aligned} \tag{C14}$$

where $\hat{\mathcal{K}}_{i'j'}$, is a well-defined term in the development of

$$\begin{aligned}\left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{+\infty} dE\, e^{-iEs''} \int_{\Gamma''} dl\, e^{-ilt''} (-1)^{i'+j'} \\ \times (\mathbf{p}''^N : \mathbf{k}''^n | \left[V^{>} (H_0^{>} - E - \tfrac{1}{2}l)^{-1} \right]^{i'} \\ \times \left[V^{<} (H_0^{<} - E + \tfrac{1}{2}l)^{-1} \right]^{j'} | \mathbf{p}'^N : \mathbf{k}^m)\end{aligned} \tag{C15}$$

Let us assume that in some i' and j', $\mathcal{K}_{i-i', j-j'}$ in Eq. (C14) decomposes to some disconnected components. Then, by the same argument as in the previous paper[14] showing the conservation law of the probability in the perturbed solution, whole summations of this type of disconnected terms cancel each other due to the "compensative relation" when we take the summation of the momenta of unfixed particles. It implies that the collision

s.operator does not contain disconnected components, i.e., all particles are connected through their interaction.

As a consequence of the above arguments regarding initial conditions, we conclude that the perturbation series for $\phi^{(m)}$ in the reduced master equation consists of m disconnected components, each of which includes one of the particles in $1, \ldots, m$.

As has been shown by Hugenholtz[25] in the perturbation expansion of $(H - z)^{-1}$ in $\mathcal{H}$, the summation of all possible terms obtained by ordering a set of disconnected components can be expressed in a simple form as a convolution integral of each component on z. For example, if $\Phi(z)$ and $\Psi(z)$ are disconnected components, the inverse Laplace transform of the summation of all ordering of $\Phi(z)$ and $\Psi(z)$ is expressed by

$$\left(\frac{1}{2\pi i}\right)^2 \int_\Gamma dz\, e^{-iz[t+(1/2)s]} \int_c d\xi\, \Phi(z-\xi)\Psi(\xi) \tag{C16}$$

where the path of integration c is a contour encircling all singular points of the integrand on the real axis in the positive direction, but not encircling the singular points located on the straight line through z parallel to the real axis.

This theorem can be extended easily to the product of two resolvent $R^{>}(z)R^{<}(z')$ which depends on two complex variables, z and z' (see Ref. 14 for more details).

The final part of our proof is, therefore, to show that the product of the convolution integrals on z and z' can be rewritten as the convolution form on $l = z - z'$ and $E = (z + z')/2$. It can be done only by changing variables as follows:

$$\left(\frac{1}{2\pi i}\right)^4 \int_\Gamma dz\, e^{-iz[t+(1/2)s]} \int_{\Gamma'} dz'\, e^{iz'[t-(1/2)s]} \int_c d\xi \int_{c'} d\xi'\, F(z-\xi, z'-\xi')G(\xi,\xi')$$

$$= \left(\frac{1}{2\pi i}\right)^4 \int_{-\infty}^{+\infty} dE\, e^{-iEs} \int_{\Gamma''} dl\, e^{-ilt} \int_{-\infty}^{+\infty} dE' \int_{\gamma''} dl''\, F_{E-E'}(l-l')G_{E'}(l') \tag{C17}$$

for $t \geqslant \frac{1}{2}|s|$, where $F_E(l) \equiv F(E + \frac{1}{2}l, E - \frac{1}{2}l)$, and the path of γ'' is parallel to the real axis on the upper half-plane of the complex l' plane and goes to $-\infty$ from $+\infty$ with the condition that $\operatorname{Im} l' < \operatorname{Im} l$. Equation (C17) is just the Fourier–Laplace transform of the convolution integral. Thus, introducing the inverse Fourier–Laplace transform of $F_E(l)$ and $G_E(l)$ by

$$f_s(t) = \left(\frac{1}{2\pi i}\right)^2 \int_{-\infty}^{+\infty} dE\, e^{-iEs} \int_{\Gamma''} dl\, e^{-ilt} F_E(l) \tag{C18}$$

and so on, we obtain

$$(\text{C17}) = f_s(t)\, g_s(t) \tag{C19}$$

Combining all results obtained above, we conclude that the factorization property (C1) holds in the quantum kinetic equation. By putting $s = 0$ in Eq. (C1), we also have the factorization theorem for the ordinary momentum distribution function which depends on only one time parameter.

The above discussion was devoted to the system which consists of distinguishable particles. As shown previously[14] we can treat the quantum statistical system in a very similar way to the distinguishable particle system by introducing the concept of "contraction" for the Fourier coefficient. Thus, the whole concept of connectedness and disconnectedness in the perturbation series also holds for the quantum statistical system, and we can expect the proof of the factorization theorem (C1) to hold in this case by a direct extension of the argument which we have given, although the discussion is more complicated.

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